Set Theory

This document is intended to expose you to some of the terminology and symbols that we will use throughout this course. Much of this information can be found in Section 1.2 of Grinstead and Snell's *Introduction to Probability*.

1 Sets

Informally, a **set** is just a collection of objects. These could be numbers, people, physical objects, or pretty much anything else. For example, we could talk about

"the set of all Presidents of the United States"

or

"the set of all even integers."

There is actually a precise definition of a set, but we're not going to worry about it. (In general, problems can arise if one is not careful. If you're curious about this, look up *Russell's Paradox*.)

The objects that make up a set are called its **elements**. If we have a set S and a is an element of S, we write

$$\boxed{a \in S}.$$

If a is not an element of S, we instead write

$$\boxed{a \notin S}.$$

We can often define a set by simply listing all of its elements. When doing this, we usually enclose the list in braces: $\{\cdot\}$.

Example. Define

$$S = \{1, 3, 5, 7, 18\}.$$

Then S is a set, and $3 \in S$. The number $15 \notin S$, since it is not on the list that we used to define S.

Example. There is a special set $\varnothing$, called the **empty set**, which is defined to be the set which contains no elements.

Quite often it is infeasible to simply list all the elements of a set. For example, it would be impossible to list all of the even integers. In situations like this, we can describe a set by specifying that its elements should satisfy some defining property. We write

$$S = \{n : n \text{ is an even integer}\},$$

read "the set of all n such that n is an even integer." This sort of notation is called **set-builder notation**. (**Note:** Some authors use a vertical line | in place of the colon, but the meaning is the same.)

Example. In linear algebra you learned about the span of a set of vectors v_1, v_2, v_3 in $\mathbb{R}^n$. In set-builder notation, one can write

$$\text{span}(v_1, v_2, v_3) = \{a_1v_1 + a_2v_2 + a_3v_3 : a_1, a_2, a_3 \text{ are real numbers }\}.$$

2 Constructions on Sets

There are a few operations on sets which will be important to know. The first will be the notion of a subset, and the others will allow us to build new sets out of old ones.

Definition 1. *Let A and B be sets. We say that B is a* ***subset*** *of A if every element of B is also an element of A. We write*

$$\boxed{B \subset A}$$

and we will sometimes say that "A contains B."

It is entirely possible a subset B of A could actually be all of A, i.e. $B = A$. If we want to emphasize that this is not the case, we will write

$$\boxed{B \subsetneq A}$$

and say that B is a **proper** subset of A.

Example. Let

$$A = \{\text{Presidents of the United States}\},$$

and let

$$B = \{\text{George Washington, Franklin Pierce, Barack Obama}\}.$$

Then $B \subset A$, and in fact B is a proper subset of A.

Example. Let

$$S = \{1, 3, 5, 7, 18\}.$$

Then the set $T = \{1, 7\}$ is a subset of S.

Example. Let A be any set. Since the empty set $\varnothing$ contains no elements, all of its elements are contained in A. (Some would say that a statement like this is *vacuously true*.) Therefore, $\varnothing \subset A$. In other words, the empty set is contained in *any* set.

The next few constructions will allow us to construct new sets by combining two other sets in some way. They are called the union, intersection, difference, and Cartesian product.

Definition 2. *Let A and B be sets. The* ***union*** *of A and B, written*

$$\boxed{A \cup B},$$

is the set whose elements consist of all the elements of A and all the elements of B. That is,

$$\boxed{A \cup B = \{x : x \in A \text{ or } x \in B\}}.$$

Example. Let $A = \{\text{red}, \text{orange}\}$ and $B = \{\text{tiger, lion, bear}\}$. Then

$$A \cup B = \{\text{red, orange, tiger, lion, bear}\}.$$

Example. Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$. Then

$$A \cup B = \{1, 2, 3, 4, 6\}.$$

Remark. Note that A and B are always subsets of $A \cup B$. Also, if A is any set, $A \cup \varnothing = A$.

Definition 3. *The **intersection** of two sets A and B, written*

$$\boxed{A \cap B},$$

is the set whose elements are all the elements which lie in both A and B. That is,

$$\boxed{A \cap B = \{x : x \in A \text{ and } x \in B\}}.$$

Example. If $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$, then

$$A \cap B = \{2\}.$$

Example. Let $A = \{\text{even integers}\}$ and $B = \{\text{odd integers}\}$. Then

$$A \cap B = \{x : x \text{ is both an even and an odd integer}\} = \varnothing,$$

since no integer is both even and odd.

Remark. We always have $A \cap B \subset A$ and $A \cap B \subset B$. Also, if A is any set, $A \cap \varnothing = \varnothing$.

Definition 4. *The **difference** of A and B, written*

$$\boxed{A - B},$$

consists of all of the elements of A which do not lie in B. That is,

$$\boxed{A - B = \{a \in A : a \notin B\}}.$$

Example. Let $A = \{1, 2, 3, 4\}$ and $B = \{2\}$. Then

$$A - B = \{1, 3, 4\}.$$

Example. Let $A = \{\text{integers}\}$ and $B = \{\text{even integers}\}$. Then

$$A - B = \{\text{odd integers}\}.$$

Definition 5. *In probability theory, the sets we will be working with will be subsets of the sample space S. The **complement** of A with respect to S, written*

$$A^c$$

consist of all the elements of S which do not lie in A. That is,

$$\boxed{A^c = \{x \in S : x \notin A\}}.$$

Example. Let our sample space be $S = \{1, 2, 3, 4, 5, 6\}$ and $A = \{3, 4\}$. Then

$$A^c = \{1, 2, 5, 6\}.$$